Content

Tutorial 4
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BACK

Questions of today

Note:

- I add the assumption $f(0) \neq 0$ to the second question. The case f has a zero of order m can be handled by considering f/z^m (and with a factor z^m in front of the infinite product).

1. Find all entire functions which are uniformly continuous.
2. Let f be an entire function with zeroes $\{a_n\}$ and $f(0) \neq 0$. Then there exists an entire function g and a sequence such that nonnegative integers $\{p_n\}$ such that

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\Big(\frac{z}{a_n}\Big)$$

3. Let Ω be an open subset of $\mathbb{C}$. Let $\{a_n\}$ be a sequence in Ω without limit points. Show that there exists a holomorphic functions $f: \Omega \to \mathbb{C}$ whose zeroes are precisely the $\{a_n\}$.
4. (Blaschke Products) Let $D = D_1$ be the open unit disc, and let $\{a_n\}$ be a sequence of nonzero complex numbers in D. Suppose

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty.$$

   Show that the product

$$f(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \overline{a_n} z} \frac{|a_n|}{a_n}$$

   defines a holomorphic function on D whose zeroes set are exactly $\{a_n\}$.
5. Let f be an entire function of finite growth order, show that f assumes each complex value with at most one exception. (You can use the last homework in HW2 to show that if the growth order is not an integer, then f assumes each complex value an infinite number of times)
6. If f and g be entire functions of finite order λ. Let $\{a_n\}$ be a sequence such that $f(a_n) = g(a_n)$.
   - a) Suppose $\sum |a_n|^{-(\lambda+\epsilon)} = \infty$ for some $\epsilon > 0$, show that $f = g$.
   - b) Find all entire functions f of finite order such that $f(\log n) = n$.

Hints & solutions of today

1. Show that $f(z+h) = f(z)$ for all small h and all z. Hence show that $\{z : f'(z) = f'(0)\}$ has a limit point at the origin.
2. Similar as the Weierstrass theorem, we need to show that for any z,

$$\sum |1 - E_{p_n}(\frac{z}{a_n})|$$

   converges. We know, however, for each $z \in \mathbb{C}$, we have

$$\left|\frac{z}{a_n}\right| < \frac{1}{2}$$

   except finitely many a_n. But we also know that

$$|1 - E_{p_n}(w)| \leq c|w|^{p_n+1}.$$

   We can then simply take $p_n = n - 1$.
3. We make some simplifications. First, if the zero set is finite, then we can use a polynomial function, so we may assume the zero set is infinite. On the other hand, if $a \in \Omega$ not inside the zero set of f. We can consider the change of variable $z \mapsto \frac{1}{z-a}$, and assume the complement of Ω is bounded, we thus need to prove the following:

   > Let Ω be the complement of a compact subset K of $\mathbb{C}$, and $\{a_n\}$ is an infinite sequence of points in Ω such that $\{a_n\}$ has no limit points in Ω and has no subsequence converging to infinity. Then there exists a holomorphic function
   >
   > $$f : \Omega \to \mathbb{C}$$
   >
   > such that the zero set of f is exactly $\{a_n\}$, and f is bounded at infinitely.

   We now prove the above statement. For each n, we choose $b_n \in K$ so that $|a_n - b_n|$ is the smallest. (i.e. $|a_n - b_n| \leq |a_n - b|$ for any $b \in K$) We then define

$$f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{a_n - b_n}{z - b_n}\right).$$

   Since E_n has a simple zero at 1, our f, if well defined, has a zero at a_n for each n. As in Question 2, we just need to show that for some zn

$$\left|\frac{a_n - b_n}{z - b_n}\right| < \frac{1}{2}$$

   for all $n > N$. (We need also the n can be chosen uniformly on compact subsets of Ω) This would follow from the following lemma:

   Lemma: $|a_n - b_n| \to 0.$

   (Proof of the lemma): If not, then by passing to a subsequence, we can find some $\epsilon > 0$ such that

$$|a_n - b_n|$$

   for all n. Let A denotes the set $\{a_n\}$, the above says that

$$\text{dist}(A, K) \geq \epsilon.$$

   Therefore, A has no limit point on K. On the other hand, the assumption says that A has no limit point in Ω. We thus know that A has no limit points in $\mathbb{C}$.
   Any bounded infinite subset of $\mathbb{C}$ has a limit point, so A must be unbounded. But this would imply that $\{a_n\}$ has a subsequence converging to the infinity, which contracdicts to the assumption. Therefore $|a_n - b_n| \to 0$.

   We only remains to show that f is bounded at infinity, but note that $\{a_n\}$ is bounded because it has no subsequence converging to the infinity. Therefore, for z large enough, we have

$$\left|\frac{a_n - b_n}{z - b_n}\right| < \frac{1}{2}$$

   for all n.
4. You may need the following estimates:

$$\begin{aligned}\left|1 - \frac{a_n - z}{1 - \overline{a_n} z} \frac{|a_n|}{a_n}\right| &= \left|\frac{a_n - |a_n|^2 z - a_n|a_n| + |a_n| z}{a_n - |a_n|^2 z}\right| \\ &= \left|\frac{(a_n + |a_n| z)}{a_n - |a_n|^2 z}\right| (1 - |a_n|)\end{aligned}$$

   and

$$\begin{aligned}\left|\frac{(a_n + |a_n| z)}{a_n - |a_n|^2 z}\right| &= \left|\frac{(\frac{a_n}{|a_n|^2} + \frac{1}{|a_n|} z)}{\frac{a_n}{|a_n|^2} - z}\right| \\ &\leq \frac{2(1 + |z|)}{1 - |z|}\end{aligned}$$

   for $\frac{1}{2} < |a_n| < 1$.
5. If f has no zero, then $f = \exp(g)$ from some polynomial g. Then apply fundamental theorem of algebra.
6. For part b), show that for any positive integer k, the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^k}$$

   diverges.
